ALMOST COMPLEX STRUCTURES
ON $(n-1)$-CONNECTED $2n$-MANIFOLDS

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ABSTRACT. Let M be a closed $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geq 3$. In terms of the system of invariants for such manifolds introduced by Wall, we obtain necessary and sufficient conditions for M to admit an almost complex structure.

1. INTRODUCTION

First we introduce some notations. For a topological space X, let $Vect_C(X)$ (resp. $Vect_R(X)$) be the set of isomorphic classes of complex (resp. real) vector bundles on X, and let $r: Vect_C(X) \to Vect_R(X)$ be the real reduction, which induces the real reduction homomorphism $\tilde{r}: \widetilde{K}(X) \to \widetilde{KO}(X)$ from the reduced KU-group to the reduced KO-group of X. For a map $f: X \to Y$ between topological spaces X and Y, denote by $f_u^*: \widetilde{K}(Y) \to \widetilde{K}(X)$ and $f_o^*: \widetilde{KO}(Y) \to \widetilde{KO}(X)$ the induced homomorphisms.

Let M be a $2n$-dimensional smooth manifold with tangent bundle TM. We say that M admits an *almost complex structure* (resp. a *stable almost complex structure*) if $TM \in \text{Im}\,r$ (resp. $TM \in \text{Im}\,\tilde{r}$). Clearly, M admits an almost complex structure implies that M admits a stable almost complex structure. It is a classical topic in geometry to determine which M admits an almost complex structure. See for instance [15, 5, 9, 13]. In this paper we determine those closed $(n-1)$-connected $2n$-dimensional smooth manifolds M with $n \geq 3$ that admit an almost complex structure.

Throughout this paper, M will be a closed oriented $(n-1)$-connected $2n$- dimensional smooth manifold with $n \geq 3$. In [14], C.T.C. Wall assigned to each M a system of invariants as follows.

1) $H = H^n(M; \mathbb{Z}) \cong Hom(H_n(M; \mathbb{Z}); \mathbb{Z}) \cong \oplus_{j=1}^k \mathbb{Z}$, the cohomology group of M, with k the n-th Betti number of M,

2) $I\colon H \times H \to \mathbb{Z}$, the intersection form of M which is unimodular and n- symmetric, defined by

$$I(x, y) = < x \cup y, [M] >,$$

where the homology class $[M]$ is the orientation class of M,

3) A map $\alpha\colon H_n(M; \mathbb{Z}) \to \pi_{n-1}(SO_n)$ that assigns each element $x \in H_n(M; \mathbb{Z})$ to the characteristic map $\alpha(x)$ for the normal bundle of the embedded n-sphere S_x^n representing x.

These invariants satisfy the relation ([14, Lemma 2])

$$(1.1) \qquad\qquad \alpha(x + y) = \alpha(x) + \alpha(y) + I(x, y)\, \partial\iota_n,$$

where ∂ is the boundary homomorphism in the exact sequence

$$(1.2) \qquad \cdots \to \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SO_n) \xrightarrow{S} \pi_{n-1}(SO_{n+1}) \to \cdots$$

of the fiber bundle $SO_n \hookrightarrow SO_{n+1} \to S^n$, and $\iota_n \in \pi_n(S^n)$ is the class of the identity map.

Denote by $\chi = S \circ \alpha\colon H_n(M; \mathbb{Z}) \to \pi_{n-1}(SO_{n+1}) \cong \widetilde{KO}(S^n)$ the composition map, then from (1.1) and (1.2)

$$(1.3) \qquad \chi = S \circ \alpha \in H^n(M; \widetilde{KO}(S^n)) = Hom(H_n(M; \mathbb{Z}); \widetilde{KO}(S^n))$$

can be viewed as an n-dimensional cohomology class of M, with coefficient in $\widetilde{KO}(S^n)$. It follows from Kervaire [8, Lemma 1.1] and Hirzebruch index Theorem [7] that the Pontrjagin classes $p_j(M) \in H^{4j}(M; \mathbb{Z})$ of M can be expressed in terms of the cohomology class χ and the index τ of the intersection form I (when n is even) as follows (cf. Wall [14, p. 179-180]).

Lemma 1.1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geq 3$. Then*

$$p_j(M) = \begin{cases} \pm a_{n/4}(n/2 - 1)!\chi, & n \equiv 0 \pmod 4,\ j = n/4, \\[2mm] \frac{a_{n/4}^2}{2}((n/2 - 1)!)^2\{1 - \frac{(2^{n/2-1}-1)^2}{2^{n-1}-1}\binom{n}{n/2}\frac{B_{n/4}^2}{B_{n/2}}\}I(\chi,\chi) \\[1mm] \quad + \frac{n!}{2^n(2^{n-1}-1)B_{n/2}}\tau, & n \equiv 0 \pmod 4,\ j = n/2, \\[2mm] \frac{n!}{2^n(2^{n-1}-1)B_{n/2}}\tau, & n \equiv 2 \pmod 4,\ j = n/2, \end{cases}$$

where

$$a_{n/4} = \begin{cases} 1, & n \equiv 0 \pmod 8, \\ 2, & n \equiv 4 \pmod 8, \end{cases}$$

B_m *is the m-th Bernoulli number.*

Now we can state the main results as follows.

Theorem 1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geq 3$, χ be the cohomology class defined in (1.3), τ the index of the intersection form I (when n is even). Then the necessary and sufficient conditions for M to admit a stable almost complex structure are:*

1) $n \equiv 2,\ 3,\ 5,\ 6,\ 7\ (mod\ 8)$, or
2) if $n \equiv 0\ (mod\ 8)$: $\chi \equiv 0\ (mod\ 2)$ and $\frac{(B_{n/2}-B_{n/4})}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n} \equiv 0\ (mod\ 2)$,
3) if $n \equiv 4\ (mod\ 8)$: $\frac{(B_{n/2}+B_{n/4})}{B_{n/2}B_{n/4}} \cdot \frac{\tau}{2^{n-2}} \equiv 0\ (mod\ 2)$,
4) if $n \equiv 1\ (mod\ 8)$: $\chi = 0$.

Theorem 2. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geq 3$, k be the n-th Betti number, I be the intersection form, and $p_j(M)$ be the Pontrjagin class of M as in Lemma 1.1. Then M admits an almost complex structure if and only if M admits a stable almost complex structure and one of the following conditions are satisfied:*

1) If $n \equiv 0\ (mod\ 4)$: $4p_{n/2}(M) - I(p_{n/4}(M),\ p_{n/4}(M)) = 8\ (k+2)$,
2) if $n \equiv 2\ (mod\ 8)$: there exists an element $x \in H^n(M;\mathbb{Z})$ such that
   $x \equiv \chi\ (mod\ 2)$ and $I(x,x) = (2(k+2) + p_{n/2}(M))/((n/2-1)!)^2$,
3) if $n \equiv 6\ (mod\ 8)$: there exists an element $x \in H^n(M;\mathbb{Z})$ such that
   $I(x,x) = (2(k+2) + p_{n/2}(M))/((n/2-1)!)^2$,
4) if $n \equiv 1\ (mod\ 4)$: $2((n-1)!) \mid (2-k)$,
5) if $n \equiv 3\ (mod\ 4)$: $(n-1)! \mid (2-k)$.

Remark 1.2. i) Since the rational numbers $\frac{(B_{n/2}-B_{n/4})}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n}$ and $\frac{(B_{n/2}+B_{n/4})}{B_{n/2}B_{n/4}} \cdot \frac{\tau}{2^{n-2}}$ in Theorem 1 can be viewed as 2-adic integers (see the proof of Theorem 1), it makes sense to take congruent classes modulo 2.

ii) In the cases 2) and 3) of Theorem 2, when the conditions are satisfied, the almost complex structure on M depends on the choice of x.

This paper is arranged as follows. In §2 we obtain presentations for the groups $\widetilde{KO}(M)$, $\widetilde{K}(M)$ and determine the real reduction $\tilde{r} \colon \widetilde{K}(M) \to \widetilde{KO}(M)$ accordingly. In §3 we determine the expression of $TM \in \widetilde{KO}(M)$ with respect to the presentation of $\widetilde{KO}(M)$ obtained in §2. With these preliminary results, Theorem 1 and Theorem 2 are established in §4.

2. The real reduction $\tilde{r} \colon \widetilde{K}(M) \to \widetilde{KO}(M)$

According to Wall [14], M is homotopic to a *CW* complex $(\vee_{\lambda=1}^{k} S_{\lambda}^{n}) \cup_f \mathbb{D}^{2n}$, where k is the n-th Betti number of M, $\vee_{\lambda=1}^{k} S_{\lambda}^{n}$ is the wedge sum of n-spheres which is the n-skeleton of M and $f \in \pi_{2n-1}(\vee_{\lambda=1}^{k} S_{\lambda}^{n})$ is the attaching map of $\mathbb{D}^{2n}$ which is determined by the intersection form I and the map α (cf. Duan and Wang [4]).

Let $i \colon \vee_{\lambda=1}^{k} S_{\lambda}^{n} \to M$ be the inclusion map of the n-skeleton of M and $p \colon M \to S^{2n}$ be the map collapsing the n-skeleton $\vee_{\lambda=1}^{k} S_{\lambda}^{n}$ to the base point. Then by the naturality of the Puppe sequence, we have the following exact ladder:

$$\widetilde{K}(\vee_{\lambda=1}^{k}S_{\lambda}^{n+1}) \xrightarrow{\Sigma f_{u}^{*}} \widetilde{K}(S^{2n}) \xrightarrow{p_{u}^{*}} \widetilde{K}(M) \xrightarrow{i_{u}^{*}} \widetilde{K}(\vee_{\lambda=1}^{k}S_{\lambda}^{n}) \xrightarrow{f_{u}^{*}} \widetilde{K}(S^{2n-1})$$

$$(2.1) \qquad \tilde{r} \downarrow \qquad\qquad \tilde{r} \downarrow \qquad\qquad \tilde{r} \downarrow \qquad\qquad \tilde{r} \downarrow \qquad\qquad \tilde{r} \downarrow$$

$$\widetilde{KO}(\vee_{\lambda=1}^{k}S_{\lambda}^{n+1}) \xrightarrow{\Sigma f_{o}^{*}} \widetilde{KO}(S^{2n}) \xrightarrow{p_{o}^{*}} \widetilde{KO}(M) \xrightarrow{i_{o}^{*}} \widetilde{KO}(\vee_{\lambda=1}^{k}S_{\lambda}^{n}) \xrightarrow{f_{o}^{*}} \widetilde{KO}(S^{2n-1})$$

where the horizontal homomorphisms Σf_{u}^{*}, Σf_{o}^{*}, p_{u}^{*}, p_{o}^{*}, i_{u}^{*}, i_{o}^{*} and f_{u}^{*}, f_{o}^{*} are induced by Σf, p, i and f respectively, and where Σ denotes the suspension.

Let $\mathbb{Z}\beta$ (resp. $\mathbb{Z}_{2}\beta$) be the infinite cyclic group (resp. finite cyclic group of order 2) generated by β. Then the generators ω_{C}^{m} (resp. ω_{R}^{m}) of the cyclic group $\widetilde{K}(S^{m})$ (resp. $\widetilde{KO}(S^{m})$) with $m > 0$ can be so chosen such that the real reduction $\tilde{r}\colon \widetilde{K}(S^{m}) \to \widetilde{KO}(S^{m})$ can be summarized as in Table 1 (cf. Mimura and Toda [12, Theorem 6.1, p. 211]).

Table 1. Real reduction $\tilde{r}\colon \widetilde{K}(S^{m}) \to \widetilde{KO}(S^{m})$

m (mod 8)	$\widetilde{K}(S^{m})$	$\widetilde{KO}(S^{m})$	$\tilde{r}\colon \widetilde{K}(S^{m}) \to \widetilde{KO}(S^{m})$
0	$\mathbb{Z}\omega_{C}^{m}$	$\mathbb{Z}\omega_{R}^{m}$	$\tilde{r}(\omega_{C}^{m}) = 2\omega_{R}^{m}$
1	0	$\mathbb{Z}_{2}\omega_{R}^{m}$	$\tilde{r} = 0$
2	$\mathbb{Z}\omega_{C}^{m}$	$\mathbb{Z}_{2}\omega_{R}^{m}$	$\tilde{r}(\omega_{C}^{m}) = \omega_{R}^{m}$
4	$\mathbb{Z}\omega_{C}^{m}$	$\mathbb{Z}\omega_{R}^{m}$	$\tilde{r}(\omega_{C}^{m}) = \omega_{R}^{m}$
6	$\mathbb{Z}\omega_{C}^{m}$	0	$\tilde{r} = 0$
3, 5, 7	0	0	$\tilde{r} = 0$

Denoted by $t_{ju}^{*}\colon \widetilde{K}(S_{j}^{n}) \to \widetilde{K}(\vee_{\lambda=1}^{k}S_{\lambda}^{n})$ and $t_{jo}^{*}\colon \widetilde{KO}(S_{j}^{n}) \to \widetilde{KO}(\vee_{\lambda=1}^{k}S_{\lambda}^{n})$ the homomorphisms induced by $t_{j}\colon \vee_{\lambda=1}^{k}S_{\lambda}^{n} \to S_{j}^{n}$ which collapses $\vee_{\lambda\neq j}S_{\lambda}^{n}$ to the base point. Then we have:

Lemma 2.1. *Let M be a closed oriented* $(n-1)$-*connected* $2n$-*dimensional smooth manifold with* $n \geq 3$. *Then the presentations of the groups* $\widetilde{K}(M)$ *and* $\widetilde{KO}(M)$ *as well as the real reduction* $\tilde{r}\colon \widetilde{K}(M) \to \widetilde{KO}(M)$ *can be given as in Table 2.*

Table 2. Real reduction $\tilde{r}\colon \widetilde{K}(M) \to \widetilde{KO}(M)$

n (mod 8)	$\widetilde{K}(M)$	$\widetilde{KO}(M)$	$\tilde{r}\colon \widetilde{K}(M) \to \widetilde{KO}(M)$
0	$\mathbb{Z}\xi \oplus \bigoplus_{j=1}^{k} \mathbb{Z}\eta_{j}$	$\mathbb{Z}\gamma \oplus \bigoplus_{j=1}^{k} \mathbb{Z}\zeta_{j}$	$\tilde{r}(\xi) = 2\gamma$, $\tilde{r}(\eta_{j}) = 2\zeta_{j}$
1	$\mathbb{Z}\xi$	$\mathbb{Z}_{2}\gamma \oplus \bigoplus_{j=1}^{k} \mathbb{Z}_{2}\zeta_{j}$	$\tilde{r}(\xi) = \gamma$
2	$\mathbb{Z}\xi \oplus \bigoplus_{j=1}^{k} \mathbb{Z}\eta_{j}$	$\mathbb{Z}\gamma \oplus \bigoplus_{j=1}^{k} \mathbb{Z}_{2}\zeta_{j}$	$\tilde{r}(\xi) = \gamma$, $\tilde{r}(\eta_{j}) = \zeta_{j}$
4	$\mathbb{Z}\xi \oplus \bigoplus_{j=1}^{k} \mathbb{Z}\eta_{j}$	$\mathbb{Z}\gamma \oplus \bigoplus_{j=1}^{k} \mathbb{Z}\zeta_{j}$	$\tilde{r}(\xi) = 2\gamma$, $\tilde{r}(\eta_{j}) = \zeta_{j}$
5	$\mathbb{Z}\xi$	$\mathbb{Z}_{2}\gamma$	$\tilde{r}(\xi) = \gamma$
6	$\mathbb{Z}\xi \oplus \bigoplus_{j=1}^{k} \mathbb{Z}\eta_{j}$	$\mathbb{Z}\gamma$	$\tilde{r}(\xi) = \gamma$, $\tilde{r}(\eta_{j}) = 0$
3, 7	$\mathbb{Z}\xi$	0	$\tilde{r} = 0$

where the generators ξ, η_j, γ, ζ_j, $1 \le j \le k$, satisfy:

$$\begin{cases} \xi = p_u^*(\omega_C^{2n}), \ i_u^*(\eta_j) = t_{ju}^*(\omega_C^n); \\ \gamma = p_o^*(\omega_R^{2n}), \ i_o^*(\zeta_j) = t_{jo}^*(\omega_R^n). \end{cases}$$

Proof. We assert that

a) the induced homomorphisms f_u^*, f_o^*, Σf_u^* and Σf_o^* in (2.1) are trivial, moreover,

b) the short exact sequences

(i) $$0 \to \widetilde{K}(S^{2n}) \xrightarrow{p_u^*} \widetilde{K}(M) \xrightarrow{i_u^*} \widetilde{K}(\vee_{\lambda=1}^k S_\lambda^n) \to 0$$

(ii) $$0 \to \widetilde{KO}(S^{2n}) \xrightarrow{p_o^*} \widetilde{KO}(M) \xrightarrow{i_o^*} \widetilde{KO}(\vee_{\lambda=1}^k S_\lambda^n) \to 0$$

split.

Denote by $c\colon \widetilde{KO}(X) \to \widetilde{K}(X)$ the complexification. Then by (2.1), combining these assertions with the fact that $\tilde{r} \circ c = 2$, all the results in Table 2 are easily verified.

Now we prove assertions a) and b).

Firstly, by the Bott periodicity Theorem [3], we may assume that the horizontal homomorphisms Σf_u^*, Σf_o^*, p_u^*, p_o^*, i_u^*, i_o^* and f_u^*, f_o^* in (2.1) are induced by $\Sigma^9 f$, $\Sigma^8 p$, $\Sigma^8 i$ and $\Sigma^8 f$ respectively, where Σ^j denotes the j-th iterated suspension. Note that $\Sigma^9 f \in \pi_{2n+8}(\vee_{\lambda=1}^k S_\lambda^{n+9})$ and $\Sigma^8 f \in \pi_{2n+7}(\vee_{\lambda=1}^k S_\lambda^{n+8})$, and the groups $\pi_{2n+8}(\vee_{\lambda=1}^k S_\lambda^{n+9})$ and $\pi_{2n+7}(\vee_{\lambda=1}^k S_\lambda^{n+8})$ are all in their stable range, that is $\pi_{2n+8}(\vee_{\lambda=1}^k S_\lambda^{n+9}) \cong \pi_{2n+7}(\vee_{\lambda=1}^k S_\lambda^{n+8}) \cong \oplus_{\lambda=1}^k \pi_{n-1}^s$, where π_{n-1}^S is the $(n-1)$-th stable homotopy group of spheres. Thus the fact that Σf_u^* and f_u^* are trivial can be deduced easily from Table 1 and Adams [1, proposition 7.1]; the fact that Σf_o^* and f_o^* are trivial when $n \not\equiv 1$ (*mod* 8) follows from Table 1 while the fact that Σf_o^* and f_o^* are trivial when $n \equiv 1$ (*mod* 8) follows from Adams [1, proposition 7.1]. This proves assertion a).

Secondly, (i) of assertion b) is true since the abelian group $\widetilde{K}(\vee_{\lambda=1}^k S_\lambda^n)$ is free.

Finally we prove (ii) of assertion b). For the cases $n \not\equiv 1, 2$ (*mod* 8) the proof is similar to (i).

Case $n \equiv 1$ (*mod* 8). From (2.1), Table 1 and (i) we get that $\widetilde{K}(M) \cong \mathbb{Z}$ and $\widetilde{KO}(M)$ is a finite group. Therefore, for each $x \in \widetilde{KO}(M)$, we have $2x = \tilde{r} \circ c(x) = 0$, which implies (ii) of assertion b) in this case.

Case $n \equiv 2$ (*mod* 8). By (i), we may write $\widetilde{K}(M)$ as

$$\widetilde{K}(M) = \mathbb{Z}\xi \oplus \bigoplus_{j=1}^k \mathbb{Z}\eta_j,$$

where the generators ξ, η_j, $1 \le j \le k$, satisfy $\xi = p_u^*(\omega_C^{2n})$, $i_u^*(\eta_j) = t_{ju}^*(\omega_C^n)$. By Hilton-Milnor theorem [16, p. 511] we know that the group $\pi_{2n-1}(\vee_{j=1}^k S_j^n)$ can be decomposed as:

$$\pi_{2n-1}(\vee_{j=1}^k S_j^n) \cong \oplus_{j=1}^k \pi_{2n-1}(S_j^n) \oplus_{1 \le i < j \le k} \pi_{2n-1}(S_{ij}^{2n-1}),$$

where $S_{ij}^{2n-1} = S^{2n-1}$, the group $\pi_{2n-1}(S_j^n)$ is embedded in $\pi_{2n-1}(\vee_{j=1}^k S_j^n)$ by the natural inclusion, and the group $\pi_{2n-1}(S_{ij}^{2n-1})$ is embedded by composition with the Whitehead product of certain elements in $\pi_n(\vee_{j=1}^k S_j^n)$. Hence by Duan and Wang [4, Lemma 3], the attaching map f can be decomposed accordingly as:

$$f = \Sigma_{j=1}^k f_j + g,$$

where

$$f_j \in ImJ \subset \pi_{2n-1}(S^n)$$

J being the J-homomorphism and $g \in \oplus_{1 \le i < j \le k} \pi_{2n-1}(S_{ij}^{2n-1})$. Moreover, since the suspension of the Whitehead product is trivial, it follows that the homotopy group $\pi_{2n+7}(\vee_{j=1}^k S_j^{n+8})$ can be decomposed as:

$$\pi_{2n+7}(\vee_{j=1}^k S_j^{n+8}) \cong \oplus_{j=1}^k \pi_{2n+7}(S_j^{n+8}),$$

and accordingly $\Sigma^8 f$ can be decomposed as:

$$\Sigma^8 f = \oplus_{j=1}^k \Sigma^8 f_j \in \oplus_{j=1}^k \pi_{2n+7}(S_j^{n+8})$$

with

$$\Sigma^8 f_j \in ImJ \subset \pi_{2n+7}(S_j^{n+8}) \cong \pi_{n-1}^s.$$

Denote by $e_C(\Sigma^8 f_j)$ the e_C invariant of $\Sigma^8 f_j$ defined in Adams [1], $\Psi_C^{-1} \colon \widetilde{K}(M) \to \widetilde{K}(M)$ and $\Psi_R^{-1} = id \colon \widetilde{KO}(M) \to \widetilde{KO}(M)$ the Adams operations, where id is the identity map. Then it follows from Adams [1, Proposition 7.19] that

$$e_C(\Sigma^8 f_j) = 0,$$

for each $1 \le j \le k$. Hence, by considering the map

$$\tilde{t}_j \colon (\vee_{\lambda=1}^k S_\lambda^{n+8}) \cup_{\Sigma^8 f} \mathbb{D}^{2n+8} \to S_j^{n+8} \cup_{\Sigma^8 f_j} \mathbb{D}^{2n+8}$$

which collapses $\vee_{\lambda \ne j} S_\lambda^{n+8}$ to a point, it's easy to see from [1, proposition 7.5, Proposition 7.8] and the naturality of Adams operation that

$$\Psi_C^{-1}(\eta_j) = (-1)^{n/2}\eta_j + l \cdot ((-1)^n - (-1)^{n/2})\xi \in \widetilde{K}(M)$$

for each η_j, and for some $l \in \mathbb{Z}$. Therefore from

$$\tilde{r} \circ \Psi_C^{-1} = \Psi_R^{-1} \circ \tilde{r},$$

we have

$$\Psi_R^{-1} \tilde{r}(\eta_j) = -\tilde{r}(\eta_j) + 2l\tilde{r}(\xi).$$

That is

$$2\tilde{r}(\eta_j - l\xi) = 0.$$

But from (2.1) and Table 1, we get

$$i_o^* \tilde{r}(\eta_j - l\xi) = t_{oj}^*(\omega_R^n).$$

That is

$$\tilde{r}(\eta_j - l\xi) \neq 0 \in \widetilde{KO}(M).$$

Thus (ii) of assertion b) in this case is established and the proof is finished.   □

Remark 2.2. Since the induced homomorphisms $i^* \colon H^n(M;\mathbb{Z}) \to H^n(\vee_{\lambda=1}^k S_\lambda^n;\mathbb{Z})$ and $p^* \colon H^{2n}(S^{2n};\mathbb{Z}) \to H^{2n}(M;\mathbb{Z})$ are both isomorphisms, and the generator $\omega_C^{2n} \in \widetilde{K}(S^{2n})$ can be chosen such that its n-th chern class $c_n(\omega_C^{2n}) = (n-1)!$(cf. Hatcher [6, p. 101]), from the naturality of the chern class, we get

$$c_i(\xi) = \begin{cases} (n-1)!, & i = n, \\ 0, & \text{others.} \end{cases}$$

Similarly, when n is even, η_j, $1 \leq j \leq k$, can be chosen such that

$$c_{n/2}(\Sigma_{j=1}^k x_j \eta_j) = (n/2-1)!(x_1, x_2, ..., x_k) \in H^n(M;\mathbb{Z}),$$

where $x_j \in \mathbb{Z}$ for all $1 \leq j \leq k$ (since $H^n(M;\mathbb{Z}) \cong \oplus_{j=1}^k \mathbb{Z}$, we can write an element $x \in H^n(M;\mathbb{Z})$, under the isomorphism i^*, as the form $(x_1, x_2, ..., x_k)$).

Remark 2.3. As in Remark 2.2, if we write χ as $(\chi_1, ..., \chi_k) \in H^n(M;\widetilde{KO}(S^n))$, where

$$\chi_j \in \widetilde{KO}(S^n) \cong \begin{cases} \mathbb{Z}, & n \equiv 0 \ (mod\ 4), \\ \mathbb{Z}_2, & n \equiv 1, 2 \ (mod\ 8), \\ 0, & \text{others,} \end{cases}$$

then since the tangent bundle of sphere is stably trivial, it follows that

$$i_o^*(TM) = \Sigma_{j=1}^k \chi_j t_{jo}^*(\omega_R^n).$$

3. The tangent bundle of M

Denote by $\dim_c \alpha$ the dimension of $\alpha \in Vect_C(M)$. When $n \equiv 0 \ (mod\ 4)$, we set

$$\hat{A}(M) = <\hat{\mathfrak{A}}(M), [M]>,$$
$$\hat{A}_C(M) = <ch(TM \otimes \mathbb{C}) \cdot \hat{\mathfrak{A}}(M), [M]>,$$
$$\hat{A}_\chi(M) = <ch(\Sigma_{j=1}^k \chi_j \eta_j) \cdot \hat{\mathfrak{A}}(M), [M]>,$$

where ch denotes the chern character, and $\hat{\mathfrak{A}}(M)$ is the $\mathfrak{A}$-class of M (cf. Atiyah and Hirzebruch [2]). It follows from the differentiable Riemann-Roch theorem (cf. Atiyah and Hirzebruch [2]) that $\hat{A}(M)$, $\hat{A}_C(M)$ and $\hat{A}_\chi(M)$ are all integers. In particular, $\hat{A}_\chi(M)$ is even when $\chi \equiv 0 \ (mod\ 2)$.

Using the notation above, we get

Lemma 3.1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geq 3$. Then TM can be expressed by the generators γ, ζ_j, $1 \leq j \leq$*

k of $\widetilde{KO}(M)$ as follows:

$$TM = \begin{cases} l\gamma + \Sigma_{j=1}^k \chi_j \zeta_j, & n \equiv 0, 2, 4 \ (mod \ 8), \\ l\gamma, & n \equiv 6 \ (mod \ 8), \\ \Sigma_{j=1}^k \chi_j \zeta_j, & n \equiv 1 \ (mod \ 8), \\ 0, & n \equiv 3, 5, 7 \ (mod \ 8), \end{cases}$$

where

$$l = \begin{cases} \hat{A}_C(M) + (\Sigma_{j=1}^k a_{n/4}\chi_j \dim_c \eta_j - 2n)\hat{A}(M) - a_{n/4}\hat{A}_\chi(M), & n \equiv 0 \ (mod \ 4), \\ -\frac{p_{n/2}(M)}{2((n-1)!)}, & n \equiv 2 \ (mod \ 4). \end{cases}$$

Proof. Case $n \equiv 0 \ (mod \ 8)$. By Remark 2.3, we may suppose that

$$TM = l\gamma + \Sigma_{j=1}^k \chi_j \zeta_j \in \widetilde{KO}(M),$$

where $l \in \mathbb{Z}$. Hence from $\tilde{r} \circ c = 2$ and Table 2, we have

(3.1) $c(TM) = TM \otimes \mathbb{C}$

$$= l\xi + \Sigma_{j=1}^k \chi_j \eta_j \in \widetilde{K}(M).$$

Now if we regard ξ and χ_j as complex vector bundles, then from (3.1) we have

$$TM \otimes \mathbb{C} \oplus \varepsilon^s \cong l\xi \oplus \bigoplus_{j=1}^k \chi_j \eta_j \oplus \varepsilon^t,$$

for some $s, \ t \in \mathbb{Z}$ satisfying

$$s - t \ = \ l \cdot \dim_c \xi + \Sigma_{j=1}^k \chi_j \dim_c \eta_j - 2n,$$

where ε^j is the trivial complex vector bundle of dimension j. Thus we have

$$\hat{A}_C(M) = - (l \cdot \dim_c \xi + \Sigma_{j=1}^k \chi_j \dim_c \eta_j - 2n)\hat{A}(M)$$
$$+ < ch(l\xi + \Sigma_{j=1}^k \chi_j \eta_j) \cdot \hat{\mathfrak{A}}(M), [M] >,$$

that is

$$l = \hat{A}_C(M) + (\Sigma_{j=1}^k \chi_j \dim_c \eta_j - 2n)\hat{A}(M) - \hat{A}_\chi(M).$$

Cases $n \equiv 2, 4, 6 \ (mod \ 8)$ can be proved by the same way as above. Note that in the case $n \equiv 2 \ (mod \ 4)$ the calculation of $\hat{A}_C(M)$ is replaced by the calculation of the n-th chern class of $TM \otimes \mathbb{C}$.

Case $n \equiv 1 \ (mod \ 4)$. From Milnor and Kervaire [10, Lemma 1] and Adams [1, Theorem 1.3], we get that $\chi = 0$ implies $TM = 0 \in \widetilde{KO}(M)$. Then

i) case $n \equiv 5 \ (mod \ 8)$. $TM = 0 \in \widetilde{KO}(M)$ because $\chi = 0$ in this case.

ii) case $n \equiv 1 \ (mod \ 8)$. By Remark 2.3, we may suppose that

$$TM = l\gamma + \Sigma_{j=1}^k \chi_j \zeta_j,$$

where $l \in \mathbb{Z}_2$. Then if $\chi = 0$, we have $l = 0$ because $TM = 0$. If $\chi \neq 0$ and $l \neq 0$, suppose that $\chi_\lambda \neq 0$ for some $1 \leq \lambda \leq k$, set

$$\zeta'_j = \begin{cases} \zeta_j & \text{if } j \neq \lambda, \\ \zeta_j + \gamma & \text{if } j = \lambda. \end{cases}$$

Hence γ, ζ'_j, $1 \leq j \leq k$, which satisfy the conditions in Lemma 2.1, are also the generators of $\widetilde{KO}(M)$, and we have $TM = \Sigma_{j=1}^k \chi_j \zeta'_j$. This implies that the generators γ, ζ_j, $1 \leq j \leq k$, of $\widetilde{KO}(M)$ in Lemma 2.1 can always be chosen such that $TM = \Sigma_{j=1}^k \chi_j \zeta_j$.

Case $n \equiv 3 \ (mod\ 4)$. $TM = 0$ because $\widetilde{KO}(M) = 0$ in this case. $\qquad \square$

4. Almost complex structure on M

We are now ready to prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Cases 1) and 2) $n \equiv 0 \ (mod\ 4)$. In these cases, we get that (cf. Wall [14, p. 179-180])

$$\hat{A}(M) = -\frac{B_{n/2}}{2(n!)} p_{n/2}(M) + \frac{1}{2}\{\frac{B_{n/4}^2}{4((n/2)!)^2} + \frac{B_{n/2}}{2(n!)}\} I(p_{n/4}(M), p_{n/4}(M)),$$

$$\hat{\mathfrak{A}}(M) = 1 - \frac{B_{n/4}}{2((n/2)!)} p_{n/4}(M) + \hat{A}(M),$$

$$ch(TM \otimes \mathbb{C}) = 2n + (-1)^{n/4+1} \frac{p_{n/4}(M)}{(n/2-1)!} + \frac{I(p_{n/4}(M), p_{n/4}(M)) - 2p_{n/2}(M)}{2((n-1)!)}.$$

Hence by Lemma 1.1 we have

$$(4.1) \qquad \hat{A}_C(M) = 2n\{1 + \frac{1}{B_{n/2}} + \frac{(2^{n-1}-1)}{(2^{n/2}-1)^2} \cdot \frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}}\}\hat{A}(M)$$

$$+ \frac{1}{(2^{n/2}-1)^2} \cdot \frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n}.$$

Moreover since the denominator of B_m, when written as the most simple fraction, is always square free and divisible by 2 (cf. Milnor [11, p. 284]), we may set $B_m = b_m/(2c_m)$, where c_m and b_m are odd integers. Then multiply each side of (4.1) by $(2^{n/2}-1)^2 \cdot b_{n/2} \cdot b_{n/4}$, we get that

$$(2^{n/2}-1)^2 b_{n/2} b_{n/4} \hat{A}_C(M) = 2n\{(2^{n/2}-1)^2 \cdot b_{n/2} \cdot b_{n/4} + 2(2^{n/2}-1)^2 b_{n/4} c_{n/2}$$

$$+ 2(2^{n-1}-1)((-1)^{n/4}b_{n/2}c_{n/4} - b_{n/4}c_{n/2})\}\hat{A}(M)$$

$$+ 2((-1)^{n/4}b_{n/2}c_{n/4} - b_{n/4}c_{n/2})\frac{n\tau}{2^n}.$$

Since $\hat{A}_C(M)$ and $\hat{A}(M)$ are integers and $(2^{n/2} - 1)^2 \cdot b_{n/2} \cdot b_{n/4}$ is an odd integer, it follows that

$$\frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n}$$

is a 2-adic integer, and hence

$$\hat{A}_C(M) \equiv 0 \ (mod\ 2) \iff \frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n} \equiv 0 \ (mod\ 2).$$

Then by combining these facts with Lemma 2.1 and Lemma 3.1, one verifies the results in these cases.

Cases 3) and 4) $n \not\equiv 0 \ (mod\ 4)$ can be deduced easily from Lemma 2.1 and Lemma 3.1.   □

To prove Theorem 2, we need the following lemma (see Sutherland [13] for the proof).

Lemma 4.1. *Let N be a closed smooth $2n$-manifold. Then N admits an almost complex structure if and only if it admits a stable almost complex structure α satisfying $c_n(\alpha) = e(N)$, where $e(N)$ is the Euler class of N.*

Proof of Theorem 2. Firstly, it follows from Lemma 4.1 that M admits an almost complex structure if and only if there exists an element $\alpha \in \widetilde{K}(M)$ such that

$$(4.2) \qquad \begin{cases} \tilde{r}(\alpha) = TM \in \widetilde{KO}(M), \\ c_n(\alpha) = e(M). \end{cases}$$

Secondly, if there exists an element $\alpha \in \widetilde{K}(M)$ such that $\tilde{r}(\alpha) = TM \in \widetilde{KO}(M)$, then we have the following identity (cf. Milnor [11, p. 177]):

$$(4.3) \qquad \left(\sum_j (-1)^j c_j(\alpha)\right) \cdot \left(\sum_j c_j(\alpha)\right) = \sum_j (-1)^j p_j(M).$$

Now we prove Theorem 2 case by case.

Case 1) $n \equiv 0 \ (mod\ 4)$. In this case $e(M) = k + 2$. From Lemma 4.1 we know that M admits an almost complex structure if and only if there exists an element $\alpha \in \widetilde{K}(M)$ such that (4.2) is satisfied. Now (4.3) becomes

$$(1 + c_{n/2}(\alpha) + c_n(\alpha)) \cdot (1 + c_{n/2}(\alpha) + c_n(\alpha)) = 1 + (-1)^{n/4}p_{n/4}(M) + p_{n/2}(M),$$

it follows that

$$c_{n/2}(\alpha) = (-1)^{n/4}\frac{1}{2}p_{n/4}(M),$$

hence

$$c_n(\alpha) = \frac{1}{2}p_{n/2}(M) - \frac{1}{8}I(p_{n/4}(M),\ p_{n/4}(M)).$$

Therefore from (4.2) we get that, M admits an almost complex structure if and only if M admits a stable almost complex structure and satisfies

$$4p_{n/2}(M) - I(p_{n/4}(M), p_{n/4}(M)) = 8(k+2).$$

Case 2) $n \equiv 2 \pmod 8$. In this case $e(M) = k+2$. Set $\alpha = l\xi + \Sigma_{j=1}^{k} x_j \eta_j \in \widetilde{K}(M)$ where $l \in \mathbb{Z}$ is the integer as in Lemma 3.1 and $x_j \in \mathbb{Z}$, such that $x_j \equiv \chi_j \pmod 2$. Then from Lemma 2.1 and Lemma 3.1, we know that $\tilde{r}(\alpha) = TM \in \widetilde{KO}(M)$. Hence by (4.2), we see that M admits an almost complex structure if and only if

$$\begin{cases} \alpha = l\xi + \Sigma_{j=1}^{k} x_j \eta_j \in \widetilde{K}(M), \\ c_n(\alpha) = e(M). \end{cases}$$

Let $x = (x_1, x_2, ..., x_k) \in H^n(M; \mathbb{Z})$. Then by Remark 2.2

$$c_{n/2}(\alpha) = (n/2 - 1)! x.$$

Now (4.3) is

$$(1 - c_{n/2}(\alpha) + c_n(\alpha)) \cdot (1 + c_{n/2}(\alpha) + c_n(\alpha)) = 1 - p_{n/2}(M),$$

therefore

$$\begin{aligned} c_n(\alpha) &= \frac{1}{2}(I(c_{n/2}(\alpha), c_{n/2}(\alpha)) - p_{n/2}(M)) \\ &= \frac{1}{2}\{((n/2 - 1)!)^2 I(x, x) - p_{n/2}(M)\}. \end{aligned}$$

Thus it follows from (4.2) that M admits an almost complex structure if and only if there exists an element $x \in H^n(M; \mathbb{Z})$ such that

$$\begin{cases} x \equiv \chi \pmod 2, \\ I(x, x) = (2(k+2) + p_{n/2}(M))/((n/2 - 1)!)^2. \end{cases}$$

Case 3) $n \equiv 6 \pmod 8$. The proof is similar to the proof of case 2).

Case 4) $n \equiv 1 \pmod 4$. Now $e(M) = 2 - k$. From (4.2), Lemma 2.1, Lemma 3.1 and Remark 2.2, we see that M admits an almost complex structure if and only if

$$\begin{cases} \chi = 0, \\ \alpha = 2a\xi, \\ 2a(n-1)! = 2 - k, \end{cases}$$

where $a \in \mathbb{Z}$. Hence by Lemma 3.1 and Lemma 2.1, M admits an almost complex structure if and only if M admits a stable almost complex structure and

$$2(n-1)! \mid (2 - k).$$

Case 5) $n \equiv 3 \pmod 4$. The proof is similar to the proof of case 4). $\square$

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